

Proof:

$$\begin{aligned}
 \nabla \cdot (\mathbf{F} \wedge \mathbf{G}) &= \sum \left\{ \mathbf{i} \cdot \frac{\partial}{\partial x} (\mathbf{F} \wedge \mathbf{G}) \right\} \\
 &= \sum \left\{ \mathbf{i} \cdot \left[\mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial \mathbf{F}}{\partial x} \wedge \mathbf{G} \right] \right\} \\
 &= \sum \left[\mathbf{i}, \mathbf{F}, \frac{\partial \mathbf{G}}{\partial x} \right] + \sum \left[\mathbf{i}, \frac{\partial \mathbf{F}}{\partial x}, \mathbf{G} \right] \\
 &= - \sum \left[\mathbf{F}, \mathbf{i}, \frac{\partial \mathbf{G}}{\partial x} \right] + \sum \left[\left(\mathbf{i} \wedge \frac{\partial \mathbf{F}}{\partial x} \right) \cdot \mathbf{G} \right] \\
 &= - \mathbf{F} \cdot \sum \left(\mathbf{i} \wedge \frac{\partial \mathbf{G}}{\partial x} \right) + \left\{ \sum \left(\mathbf{i} \wedge \frac{\partial \mathbf{F}}{\partial x} \right) \right\} \cdot \mathbf{G} \\
 &= \underline{- \mathbf{F} \cdot \text{curl } \mathbf{G} + \mathbf{G} \cdot \text{curl } \mathbf{F}}.
 \end{aligned}$$

$$(v) \text{ curl } (\mathbf{F} \wedge \mathbf{G}) \equiv (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} + \mathbf{F} \text{ div } \mathbf{G} - \mathbf{G} \text{ div } \mathbf{F}.$$

Proof:

$$\begin{aligned}
 \nabla \wedge (\mathbf{F} \wedge \mathbf{G}) &= \sum \left\{ \mathbf{i} \wedge \frac{\partial}{\partial x} (\mathbf{F} \wedge \mathbf{G}) \right\} \\
 &= \sum \left\{ \mathbf{i} \wedge \left(\mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial \mathbf{F}}{\partial x} \wedge \mathbf{G} \right) \right\} \\
 &= \sum \left\{ \mathbf{i} \wedge \left(\mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x} \right) \right\} + \sum \left\{ \mathbf{i} \wedge \left(\frac{\partial \mathbf{F}}{\partial x} \wedge \mathbf{G} \right) \right\} \\
 &= \sum \left\{ \left(\mathbf{i} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \mathbf{F} - (\mathbf{i} \cdot \mathbf{F}) \frac{\partial \mathbf{G}}{\partial x} \right\} + \sum \left\{ (\mathbf{i} \cdot \mathbf{G}) \frac{\partial \mathbf{F}}{\partial x} \right. \\
 &\quad \left. - \left(\mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) \mathbf{G} \right\} \\
 &= \left(\sum \mathbf{i} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \mathbf{F} - \mathbf{F} \cdot \left(\sum \mathbf{i} \frac{\partial}{\partial x} \right) \mathbf{G} + \mathbf{G} \cdot \left(\sum \mathbf{i} \frac{\partial}{\partial x} \right) \mathbf{F} \\
 &\quad - \left(\sum \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) \mathbf{G} \\
 &= \underline{(\nabla \cdot \mathbf{G}) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\nabla \cdot \mathbf{F}) \mathbf{G}}.
 \end{aligned}$$

(N.B. In this example naïve application of the vector product rule would give the wrong result:

sometimes convenient to work with other systems of orthogonal coordinates. The two most common are those afforded by *cylindrical polar coordinates* and *spherical polar coordinates*.

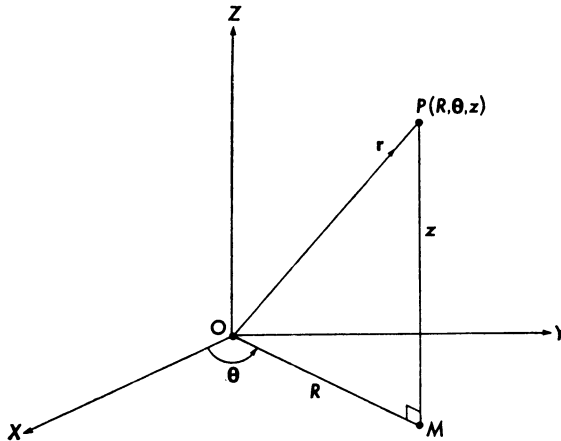


FIG. 1.24

Figure 1.24 shows a point P in a Cartesian tri-rectangular frame specified by axes OX, OY, OZ . PM is drawn perpendicularly to the coordinate plane XOY . Let $XOM = \theta$, $OM = R$, $PM = z$. Then (R, θ, z) are called the cylindrical polar coordinates of P . Knowledge of these elements clearly enables P to be determined uniquely. The three surfaces through P

$$R = \text{const.}, \quad \theta = \text{const.}, \quad z = \text{const.}$$

are respectively (i) the right circular cylinder through P , axis OX and radius R ; (ii) the plane through OZ set at angle θ to the coordinate plane ZOX ; (iii) the plane through P parallel to the coordinate plane XOY and distant z from it. Clearly these three surfaces intersect mutually orthogonally, so that the system of coordinates is an orthogonal one. We observe that if (x, y, z) are the Cartesian coordinates of P , then

$$\begin{aligned} x &= R \cos \theta, & y &= R \sin \theta, & z &= z; \\ R &= \sqrt{(x^2 + y^2)}, & \theta &= \tan^{-1}(y/x), & z &= z. \end{aligned}$$

The first system of equations expresses the Cartesian coordinates in terms of the cylindrical polars: the second system gives the cylindrical polar coordinates in terms of the Cartesians.

In Fig. 1.25, P is the point chosen in a Cartesian tri-rectangular frame of axes OX, OY, OZ , such that $OP = r$, $\angle ZOP = \theta$, $\varphi = \text{angle between planes } ZOX, ZOP$. Then (r, θ, φ) are called the spherical polar coordinates of P . Knowledge of these elements clearly enables P to be determined uniquely. The three surfaces through P

$$r = \text{const.}, \quad \theta = \text{const.}, \quad \varphi = \text{const.}$$

ρ being a scalar field and c a constant. Assuming that \mathbf{E} , \mathbf{H} , \mathbf{v} and ρ are at least twice differentiable functions of x , y , z and t , establish the following equations:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \rho + \frac{1}{c^2} \frac{\partial}{\partial t} (\rho \mathbf{v}),$$

$$\nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\frac{1}{c} \nabla \wedge (\rho \mathbf{v}),$$

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0.$$

(London Univ., Gen. B.Sc. II, 1963)

15. (i) Show that

$$\operatorname{div}(\mathbf{u} \wedge \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$$

for any two vector fields \mathbf{u} and \mathbf{v} .

If $\operatorname{curl} \mathbf{a} = \mathbf{b}$, and $\operatorname{curl} \mathbf{b} = \mathbf{a}$, show that

(a) $\operatorname{div}(\operatorname{curl} \mathbf{a} \wedge \operatorname{curl} \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$,

(b) $\nabla^2 \mathbf{a} + \mathbf{a} = \mathbf{0}$, $\nabla^2 \mathbf{b} + \mathbf{b} = \mathbf{0}$.

(ii) If θ , φ , ψ are suitable functions of position, prove that

$$\begin{aligned} \int_V (\psi \nabla \theta \cdot \nabla \varphi) dV &= \int_S (\varphi \psi \nabla \theta) \cdot d\mathbf{S} - \int_V \left\{ \varphi \nabla \cdot (\psi \nabla \theta) \right\} dV \\ &= \int_S (\theta \psi \nabla \varphi) \cdot d\mathbf{S} - \int_V \left\{ \theta \nabla \cdot (\psi \nabla \varphi) \right\} dV, \end{aligned}$$

where S is a simple closed surface bounding a volume V .

(Battersea College of Technology, Dip. Tech. I, 1964)

16. Using Green's theorem that

$$\iint_S \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) dS = \iiint_\tau (U \nabla^2 V - V \nabla^2 U) d\tau,$$

show that, if P is any point of a volume τ bounded by a surface S ,

$$4\pi V_P = \iint_S \left[\frac{1}{R} \frac{\partial V}{\partial n} - V \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right] dS - \iiint_\tau \frac{1}{R} \nabla^2 V d\tau,$$

where V is a scalar function of position satisfying conditions which should be stated, V_P is the value of V at P , and R is the distance from P to a variable point of c . Hence (i) show that

$$4\pi = - \iint_S \frac{\partial}{\partial n} \left(\frac{1}{R} \right) dS,$$

so comparing this with $\partial\phi/\partial y = -k^2x/(x^2 + y^2)$, we see that

$$f'(y) = 0$$

or

$$f(y) = \text{const.}$$

As the constant is immaterial, we may take

$$\phi(x, y) = k^2 \tan^{-1}(x/y).$$

The equipotentials are thus given by the planes

$$x = cy,$$

through the z -axis. They are appropriately intersected orthogonally by the streamlines. Figure 2.6 shows the streamlines and equipotentials.

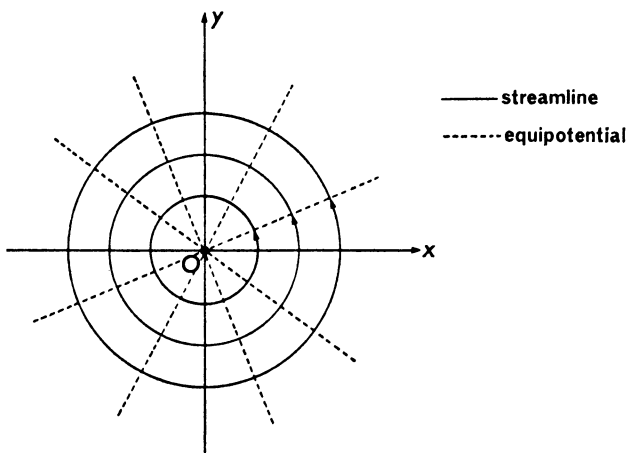


FIG. 2.6

EXAMPLE 2

For an incompressible fluid, $\mathbf{q} = [-\omega y, \omega x, 0]$ ($\omega = \text{const.}$). Discuss the nature of the flow.

We find $\nabla \cdot \mathbf{q} = 0$, so that such a flow is possible.

Further,

$$\nabla \wedge \mathbf{q} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \mathbf{k}.$$

Thus the flow is not of the potential kind. It can easily be shown that a rigid body rotating about the z -axis with constant vector angular velocity $\omega \mathbf{k}$ gives the same type of motion. (For the velocity at (x, y, z) in the body is $-\omega y \mathbf{i} + \omega x \mathbf{j}$.)

CONTENTS

PREFACE	v
SOURCES OF EXAMINATION QUESTIONS	vii
KEY TO ABBREVIATIONS	vii
1. VECTOR ANALYSIS	1
1.1 Vectors and Scalars	1
1.2 Addition and Subtraction of Vectors	2
1.3 Use of Coordinates	4
1.4 Scalar Product of Two Vectors	5
1.5 Vector Product of Two Vectors	7
1.6 Triple Products	9
1.7 Vector Moment about a Point and Scalar Moment about an Axis	10
1.8 Vector and Scalar Couples	11
1.9 Centroids	12
1.10 Differentiation of Vectors w.r.t. Scalars	13
1.11 Notion of Scalar and Vector Fields	14
1.12 The Vector Gradient and Direction Differentiator	15
1.13 Normal Flux of a Vector over a Surface; Divergence of a Vector	20
1.14 Line Integrals; Curl of a Vector Function	24
1.15 Some Vector Identities	27
1.16 Relations between Surface and Volume Integrals	32
1.16.1 Cartesian Form of Divergence Theorem	34
1.16.2 Some Theorems due to Green	36
1.16.3 Some Worked Examples	38
1.17 Relations between Line and Surface Integrals	40

1.18	Conservative Vector Fields	42
1.18.1	Conservative Fields of Force	45
1.19	General Orthogonal Curvilinear Coordinates	46
1.19.1	Arc Length in Orthogonal Coordinates	48
1.19.2	Gradient in Orthogonal Coordinates	51
1.19.3	Divergence in Orthogonal Coordinates	52
1.19.4	Laplacian in Orthogonal Coordinates	53
1.19.5	Curl of a Vector Function in Orthogonal Coordinates	53
1.19.6	Worked Examples	55
1.20	Some Cartesian Tensor Notation	58
	Exercise 1	62
2.	KINEMATICS OF FLUIDS IN MOTION	70
2.1	Real Fluids and Ideal Fluids	70
2.2	Velocity of a Fluid at a Point	71
2.3	Streamlines and Pathlines; Steady and Unsteady Flows	72
2.4	The Velocity Potential	73
2.5	The Vorticity Vector	75
2.6	Local and Particle Rates of Change	76
2.7	The Equation of Continuity	77
2.8	Worked Examples	79
2.9	Acceleration of a Fluid	84
2.10	Conditions at a Rigid Boundary	85
2.11	General Analysis of Fluid Motion	86
	Exercise 2	90
3.	EQUATIONS OF MOTION OF A FLUID	92
3.1	Pressure at a Point in a Fluid at Rest	92
3.2	Pressure at a Point in a Moving Fluid	93
3.3	Conditions at a Boundary of Two Inviscid Immiscible Fluids	95
3.4	Euler's Equations of Motion	96
3.5	Bernoulli's Equation	99
3.6	Worked Examples	100
3.7	Discussion of the Case of Steady Motion under Conservative Body Forces	103
3.8	Some Potential Theorems	104
3.9	Some Flows Involving Axial Symmetry	110
3.10	Some Special Two-Dimensional Flows	120
3.11	Impulsive Motion	121
3.12	Some Further Aspects of Vortex Motion	124
	Exercise 3	130
4.	SOME THREE-DIMENSIONAL FLOWS	136
4.1	Introduction	136
4.2	Sources, Sinks and Doublets	136

4.3	Images in a Rigid Infinite Plane	144
4.4	Images in Solid Spheres	147
4.5	Axi-Symmetric Flows; Stokes's Stream Function	150
4.5.1	Some Special Forms of the Stream Function for Axi-Symmetric Irrotational Motions	153
	Exercise 4	158
5.	SOME TWO-DIMENSIONAL FLOWS	160
5.1	Meaning of Two-Dimensional Flow	160
5.2	Use of Cylindrical Polar Coordinates	160
5.3	The Stream Function	165
5.4	The Complex Potential for Two-Dimensional, Irrotational, Incompressible Flow	167
5.5	Complex Velocity Potentials for Standard Two-Dimensional Flows	170
5.5.1	Uniform Stream	170
5.5.2	Line Sources and Line Sinks	170
5.5.3	Line Doublets	172
5.5.4	Line Vortices	174
5.6	Some Worked Examples	175
5.7	Two-Dimensional Image Systems	178
5.8	The Milne-Thomson Circle Theorem	181
5.8.1	Some Applications of the Circle Theorem	183
5.8.2	Extension of the Circle Theorem	187
5.9	The Theorem of Blasius	189
5.10	The Use of Conformal Transformation	193
5.10.1	Some Hydrodynamical Aspects of Conformal Transformation	194
5.10.2	Some Worked Examples	197
5.11	The Schwarz-Christoffel Transformation	202
5.12	Vortex Rows	207
5.12.1	Single Infinite Row of Line Vortices	207
5.12.2	The Kármán Vortex Street	208
	Exercise 5	209
6.	ELEMENTS OF THERMODYNAMICS	218
6.1	The Equation of State of a Substance	218
6.2	The First Law of Thermodynamics	219
6.3	Internal Energy of a Gas	220
6.4	Specific Heats of a Gas	221
6.5	Functions of State; Entropy	223
6.6	Maxwell's Thermodynamic Relations	224
6.7	Isothermal, Adiabatic and Isentropic Processes	229
6.8	Heat Engines, Cycle of Changes, Reversibility	230
6.9	The Carnot Cycle	231

6.10	The Second Law of Thermodynamics	234
6.11	Carnot's Theorem	235
	Exercise 6	239
7.	GAS DYNAMICS	242
7.1	Compressibility Effects in Real Fluids	242
7.2	The Elements of Wave Motion	242
7.2.1	The One-Dimensional Wave Equation	243
7.2.2	Wave Equations in Two and in Three Dimensions	245
7.2.3	Spherical Waves	246
7.2.4	Progressive and Stationary Waves	246
7.3	The Speed of Sound in a Gas	248
7.4	Equations of Motion of a Gas	249
7.5	Subsonic, Sonic and Supersonic Flows	250
7.6	Isentropic Gas Flow	253
7.7	Reservoir Discharge through a Channel of Varying Section	255
	7.7.1 Investigation of Maximum Mass Flow through a Nozzle	257
	7.7.2 Nozzle with Different Mass Flows	260
7.8	Shock Waves	264
	7.8.1 Formation of Shock Waves	264
	7.8.2 Elementary Analysis of Normal Shock Waves	268
	7.8.3 Elementary Analysis of Oblique Shock Waves	276
	7.8.4 Formation of Reflected Oblique Shocks	281
	7.8.5 Use of Shock Charts	283
	CHART <i>A</i>	<i>Facing</i> 282
	CHART <i>B</i>	<i>Facing</i> 284
7.9	The Method of Characteristics applied to Supersonic Homentropic Irrotational Gas Flows	287
	7.9.1 The Method of Characteristics for Two-Dimensional, Homentropic, Irrotational Flow	287
	7.9.2 Use of Hodograph Characteristics: Flow along a Convex Wall	294
	7.9.3 Use of Characteristic Coordinates	295
	7.9.4 Flow Round a Sharp Convex Corner: Prandtl-Meyer Expansion	298
	7.9.5 Axially-Symmetric Flows in Three Dimensions	303
	Exercise 7	305
8.	VISCOUS FLOW	310
8.1	Stress Components in a Real Fluid	310
8.2	Relations between Cartesian Components of Stress	311
8.3	Translational Motion of Fluid Element	314

CONTENTS		xiii
8.4	The Rate of Strain Quadric and Principal Stresses	315
8.5	Some Further Properties of the Rate of Strain Quadric	316
8.6	Stress Analysis in Fluid Motion	318
8.7	Relations between Stress and Rate of Strain	319
8.8	The Coefficient of Viscosity and Laminar Flow	322
8.9	The Navier-Stokes Equations of Motion of a Viscous Fluid	323
8.10	Some Solvable Problems in Viscous Flow	325
	8.10.1 Steady Motion between Parallel Planes	325
	8.10.2 Steady Flow through Tube of Uniform Circular Cross-Section	328
	8.10.3 Steady Flow between Concentric Rotating Cylinders	330
8.11	Steady Viscous Flow in Tubes of Uniform Cross-Section	332
	8.11.1 A Uniqueness Theorem	333
	8.11.2 Tube having Uniform Elliptic Cross-Section	334
	8.11.3 Tube having Equilateral Triangular Cross-Section	335
	8.11.4 Use of Harmonic Functions	336
8.12	Diffusion of Vorticity	338
8.13	Energy Dissipation due to Viscosity	339
8.14	Steady Flow past a Fixed Sphere	340
8.15	Dimensional Analysis; Reynolds Number	343
8.16	Prandtl's Boundary Layer	345
	8.16.1 Kármán's Integral Equation	348
	Exercise 8	349
9. MAGNETOHYDRODYNAMICS		354
9.1	Nature of Magnetohydrodynamics	354
9.2	Maxwell's Electromagnetic Field Equations: Medium at Rest	355
9.3	Maxwell's Electromagnetic Field Equations: Medium in Motion	356
9.4	The Equations of Motion of a Conducting Fluid	358
9.5	Rate of Flow of Charge	359
9.6	Simplification of the Electromagnetic Field Equations	360
9.7	The Magnetic Reynolds Number	361
9.8	Alfvén's Theorem	362
9.9	The Magnetic Body Force	364
9.10	Ferraro's Law of Isorotation	366
9.11	Magnetohydrostatics	368
	9.11.1 Pinch Confinement of a Plasma	369
	9.11.2 Equilibrium of Sunspots	370
9.12	Magnetohydrodynamic Waves	371
	9.12.1 A More Detailed Investigation of Alfvén Waves	373
	9.12.2 Reflection and Transmission of Alfvén Waves at a Discontinuity in Density	376

9.13	Magnetohydrodynamic Shock Waves	377
9.13.1	Shock Wave in Non-Conducting Gas with Finite Viscosity and Thermal Conductivity	377
9.13.2	MHD Effects in Shock Formation	380
9.14	Laminar Flow of a Viscous Conducting Liquid between Parallel Walls in a Transverse Magnetic Field	382
	Exercise 9	386
APPENDIX	Characteristics of Second Order Partial Differential Equations	389
SOLUTIONS		393
INDEX		397