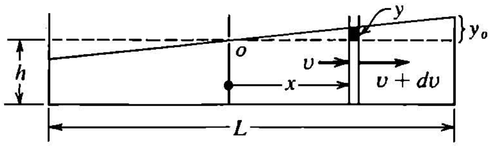
3-18 This problem is much more ambitious than the usual problems, in the sense that it requires putting together a greater number of parts. But if you tackle the various parts as suggested, you should find that they are not, individually, especially difficult, and the problem as a whole exemplifies the power of the energy-conservation method for analyzing oscillation problems.



You are no doubt familiar with the phenomenon of water sloshing about in the bathtub. The simplest motion is, to some approximation, one in which the water surface just tilts as shown but seems to remain more or less flat. A similar phenomenon occurs in lakes and is called a seiche (pronounced: saysh). Imagine a lake of rectangular cross section, as shown, of length L and with water depth h ($\ll L$). The problem resembles that of the simple pendulum, in that the kinetic energy is almost entirely due to *horizontal* flow of the water, whereas the *potential* energy depends on the very small change of vertical level. Here is a program for calculating, *approximately*, the period of the oscillations:

(a) Imagine that at some instant the water level at the extreme ends is at $\pm y_0$ with respect to the normal level. Show that the increased gravitational potential energy of the whole mass of water is given by

$$U = \tfrac{1}{6} b\rho g L y_0^2$$

where b is the width of the lake. You get this result by finding the increased potential energy of a slice a distance x from the center and integrating.

(b) Assuming that the water flow is predominantly horizontal, its speed v must vary with x, being greatest at $x = 0$ and zero at $x = \pm L/2$. Because water is incompressible (more or less) we can relate the difference of flow velocities at x and $x + dx$ to the rate of change dy/dt of the height of the water surface at x. This is a *continuity* condition. Water flows in at x at the rate vhb and flows out at $x + dx$ at the rate $(v + dv)hb$. (We are assuming $y_0 \ll h$.) The difference must be equal to $(b\,dx)(dy/dt)$, which represents the rate of increase of the volume of water contained between x and $x + dx$. Using this condition, show that

$$v(x) = v(0) - \frac{1}{hL} x^2 \frac{dy_0}{dt}$$

where

$$v(0) = \frac{L}{4h} \frac{dy_0}{dt}$$

We apply basically the same analytical technique to our N differential equations as we previously used for the two equations. We seek the normal modes; i.e., we look for sinusoidal solutions such that each particle oscillates with the same frequency. We set

$$y_p = A_p \cos \omega t \qquad (p = 1, 2, \ldots, N) \tag{5-17}$$

where A_p and ω are the amplitude and frequency of vibration of the pth particle. If we can find values of A_p and ω for which equations (5–17) satisfy the N differential equations (5–16), then we have accomplished our purpose. Note that the velocity of any particle can be obtained from equations (5–17) and is

$$\frac{dy_p}{dt} = -\omega A_p \sin \omega t \qquad (p = 1, 2, \ldots, N)$$

Thus, by choosing equations (5–17) as a trial solution, we are automatically restricting ourselves to the additional boundary condition that each particle has zero velocity at $t = 0$; i.e., each particle starts from rest.

Substituting equations (5–17) into the differential equations (5–16), we get

$$\begin{aligned}
(-\omega^2 + 2\omega_0{}^2)A_1 - \omega_0{}^2(A_2 + A_0) &= 0\\
(-\omega^2 + 2\omega_0{}^2)A_2 - \omega_0{}^2(A_3 + A_1) &= 0\\
&\vdots\\
(-\omega^2 + 2\omega_0{}^2)A_p - \omega_0{}^2(A_{p+1} + A_{p-1}) &= 0\\
&\vdots\\
(-\omega^2 + 2\omega_0{}^2)A_N - \omega_0{}^2(A_{N+1} - A_{N-1}) &= 0
\end{aligned}$$

This formidable-looking set of N simultaneous equations can be written more compactly as follows:

$$(-\omega^2 + 2\omega_0{}^2)A_p - \omega_0{}^2(A_{p-1} + A_{p+1}) = 0 \qquad (p = 1, 2, \ldots, N) \tag{5-18}$$

Our earlier boundary condition requiring the ends to be held fixed means that $A_0 = 0$ and $A_{N+1} = 0$.

The question we are asking ourselves is whether all N of these equations can be satisfied by using the same value of ω^2 in each. We saw earlier how to tackle such a problem when only two coupled oscillators were involved. The assumption that a solution existed (other than the trivial one of having all amplitudes equal to zero) led to restrictions on the ratios of the amplitudes [as expressed by equations (5–9)]. We have the same situatotal energy associated with one complete wavelength of a sinusoidal wave on a stretched string.

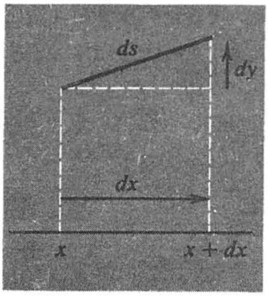


Fig. 7–18 Displacement and extension of a short segment of string carrying a transverse elastic wave.

By way of approaching this problem, we shall consider first a small segment of the string—so short that it can be regarded as effectively straight—that lies between x and $x + dx$, as shown in Fig. 7–18. We shall make the usual assumptions that the displacements of the particles in the string are strictly transverse and that the magnitude of the tension T is not changed by the deformation of the string from its normal length and configuration.

The mass of the small segment is $\mu\, dx$, and its transverse velocity (u_y) is $\partial y/\partial t$. Hence, for this segment, we have

$$\text{kinetic energy} = \tfrac{1}{2}\mu\, dx \left(\frac{\partial y}{\partial t}\right)^2$$

and we can define a kinetic energy *per unit length*—what is called the kinetic-energy *density*—for such a one-dimensional medium:

$$\text{kinetic-energy density} \equiv \frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \tag{7–31}$$

The potential energy can be calculated by finding the amount by which the string, when deformed, is longer than when it is straight. This extension, multiplied by the assumed constant tension T, is the work done in the deformation. Thus, for the segment, we have

$$\text{potential energy} = T(ds - dx)$$

where

$$\begin{aligned} ds &= (dx^2 + dy^2)^{1/2} \\ &= dx\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} \end{aligned}$$

If we assume that the transverse displacements are *small*, so that $\partial y/\partial x \ll 1$, we can approximate the above expression using the binomial expansion to two terms, thus getting

$$ds - dx \approx \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Therefore,

$$\text{potential energy} \approx \tfrac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Hence we have

$$\text{potential-energy density} \equiv \frac{dU}{dx} \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \tag{7–32}$$

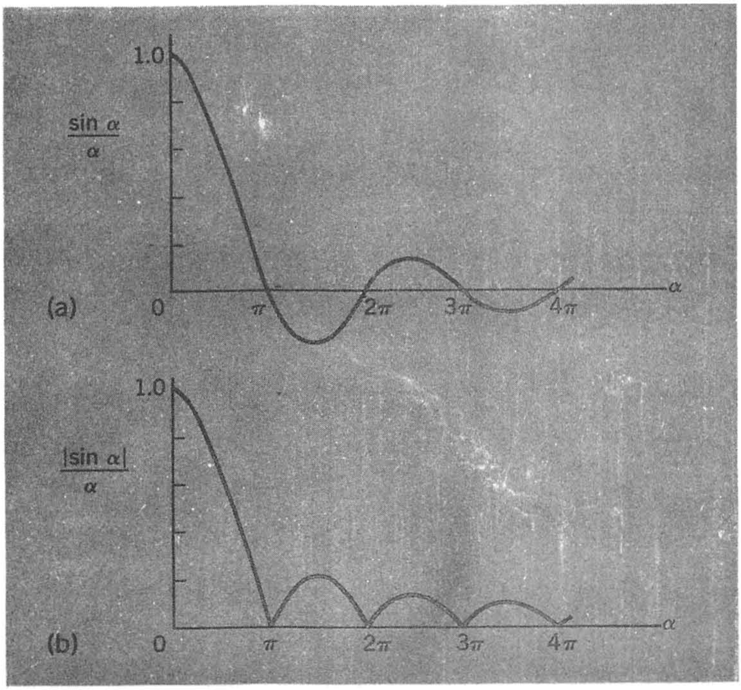


Fig. 8–24 Variation of amplitude with direction in single-slit diffraction. ($\alpha = \pi b \sin\theta/\lambda$, where θ is the direction of observation and b is the slit width.) (a) Amplitude together with phase (as shown by + or − value). (b) Absolute magnitude of amplitude.

culation. For a given value of the total phase difference φ, the vector diagram becomes a circular arc of radius R such that

$$A_0 = R\varphi$$

The resultant amplitude A under these conditions is the chord of this arc and hence is given by

$$A = 2R\sin(\varphi/2)$$

Thus we have

$$A = A_0\frac{\sin(\varphi/2)}{\varphi/2} \qquad \text{where } \frac{\varphi}{2} = \frac{\pi b \sin\theta}{\lambda} \tag{8-25}$$

This variation of resultant amplitude with direction is thus of the form $(\sin\alpha)/\alpha$, where $\alpha = \varphi/2$. This function (more formally identified as a Bessel function of order zero) has a zero whenever $\varphi/2$ is an integral multiple of π. Its general appearance is shown in Fig. 8–24(a). In Fig. 8–24(b) it is replotted without regard to sign, and its close resemblance to the amplitude curve for a diffraction grating [Fig. 8–20(b)] is then more readily appreciated.

It follows from this analysis that one slit, alone, can give rise to a diffraction pattern with a system of nodal lines, as shown in Fig. 8–25. It is essentially like the pattern around the central (zero-order) maximum of a diffraction grating, rather than a

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